
Moving a Rectangle around a Corner—Geometrically

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In [8], Moretti presented the problem of moving a couch around a corner as a challenging one for calculus students. It is interesting to solve the same problem by simple Euclidean geometry, in the same spirit as the author's solution of L'Hospital's Pulley Problem [2], namely, to stress mathematical concerns whose importance has been made increasingly evident by developments in computer science and engineering.

A main concern is the search for simplicity, which, as Dijkstra notes [4], is a prerequisite for reliability in systems design [3]. Mathematically, this means beauty and elegance of argument [5]. Arguably, geometry is simpler than calculus.

By no means does this view detract from calculus, which remains the only known approach for solving certain classes of problems and presents a unique opportunity for students to develop novel thinking abilities based on formal proof (provided, of course, that the subject is approached from this perspective).

For the problem at hand, we use the same basic technique as in [2], namely, characterizing the desired "extremal" configuration by a geometric property. Yet, whereas [2] uses force equilibrium (statics), here kinematics is the key.

Kinematics: the instantaneous center of rotation. Kinematics is the geometry of motion. Here we consider two-dimensional kinematics. If the motion involves both translation and rotation, there exists a point, rigidly linked to the body, that is momentarily at rest. The motion can be seen as pure rotation around that point, which is called the *instantaneous center of rotation* (ICR). Existence of the ICR is considered intuitively obvious in most texts on engineering mechanics (see, for example, [9]) but can also be derived from the definition as an instructive exercise. The ICR can be determined as the intersection of the lines perpendicular to the velocity vectors (or the infinitesimal displacement vectors) at two different points. For the trivial case of pure translation (of little interest here), it can be considered to be at infinity.

The ladder problem. Consider the corner depicted in Figure 1. We move the ladder by sliding endpoint A against the outer wall and the ladder itself against the inner corner C . The lines perpendicular to the velocity vectors at A and C intersect in the ICR I . The largest width used in the other corridor corresponds to the position where the endpoint B moves parallel to the second outer wall. In this position, the line perpendicular to the wall at point B must pass through the IRC I . As in [8], we write m for u/v . Using the self-explanatory *ratio chaining technique* from [2], we

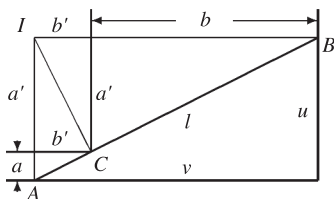


Figure 1. Ladder in extremal position.

obtain $u/v = m = a/b' = b'/a' = a'/b$. Multiplication yields $m^2 = a/a' = b'/b$ and $m^3 = a/b$. The desired length then follows from

$$l^2 = u^2 + v^2 = v^2(m^2 + 1) = (b' + b)^2(m^2 + 1) = b^2(m^2 + 1)^3 = (a^{2/3} + b^{2/3})^3.$$

The couch problem. This is illustrated by Figure 2. Adopting the notation from [8], let $w < a \leq b$. Kinematic arguments as before yield the IRC I . Clearly, $a' = a - a'' = a - w\sqrt{m^2 + 1}$ and $b' = b - b'' = b - w\sqrt{m^2 + 1}/m$. Reusing the earlier ratio chaining result, we obtain

$$m^3 = a'/b' = \frac{a - w\sqrt{m^2 + 1}}{b - w\sqrt{m^2 + 1}/m}.$$

Hence m must be a solution of

$$(bm^3 - a)^2 - w^2(m^2 - 1)^2(m^2 + 1) = 0, \quad (1)$$

so the desired length follows from

$$l^2 = u^2 + v^2 = \left(1 + \frac{1}{m^2}\right)(a + mb')^2 = \left(1 + \frac{1}{m^2}\right)(a + mb - w\sqrt{m^2 + 1})^2.$$

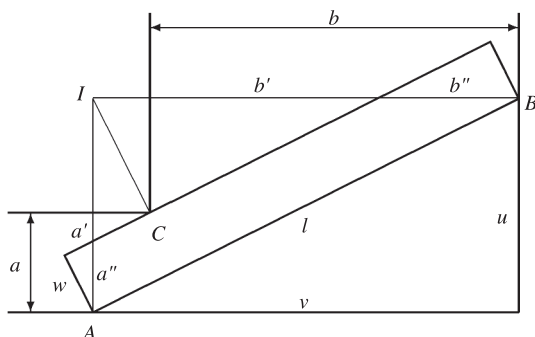


Figure 2. Couch in extremal position.

Which solutions of (1) are meaningful? Assume for physical reasons that $m > 0$. If $a = b$, then $m = 1$ is a solution. Since $a > w$ and since for $m > 0$ it is true that $(m^2 + m + 1)^2 > (m + 1)^2(m^2 + 1)$, this is the only real solution. If $a < b$, then

$$m^3 = \frac{a - w\sqrt{m^2 + 1}}{b - w\sqrt{m^2 + 1}/m}$$

requires $m < 1$. The equation

$$bm^3 - a = w(m^2 - 1)\sqrt{m^2 + 1}$$

yields $bm^3 < a$, so $0 < m < (a/b)^{1/3}$. The following argument similar to the one in [8] shows that the left-hand side of (1) has exactly one root in that interval.

The polynomial in question takes positive values for $m = 0$ and as $m \rightarrow \infty$; it is negative at $m = (a/b)^{1/3}$. By Bolzano's theorem, it has at least one real root satisfy-

ing $0 < m < (a/b)^{1/3}$ and at least one with $(a/b)^{1/3} < m$. Since the coefficients in $(b^2 - w^2)m^6 + w^2m^4 - 2abm^2 + w^2m^2 + (a^2 - w^2)$ change sign twice, there are at most two positive real roots by Descartes's Rule of Signs.

The emergence of a sixth-degree equation may be surprising at first, but perhaps less so in retrospect: triangles in ratio chaining form a (discrete) "logarithmic" spiral in which multiplication by m occurs at every stage. Pythagoras's theorem doubles the degree.

For the three-dimensional problem, an original solution and observations about irreversible mathematics in moving a sofa can be found in [1]. Although not fully satisfactory mathematically, it is highly recommended reading.

A word of caution. Solutions to the couch problem, whether using calculus [8], calculus and trigonometry [7], or pure geometry (as in this note), have in common that they somehow rely on pictures. When the position of I with respect to C in Figure 2 changes, certain signs in expressions change. Here the end result l is not affected, but the motion of the couch is different in an instructive way, with reversals of the direction in which the couch slides against corner C (the details are left as an exercise).

The need for case distinction is typical of geometrical solutions based on pictures. Sometimes case distinction yields equal results, but it may also turn out differently, and oversights have been reported [6]. In the absence of general a priori decision criteria or case-insensitive methods, caution is recommended.

REFERENCES

1. D. Adams, *Dirk Gently's Holistic Detective Agency*, Pan Books, London, 1987.
2. R. Boute, Simple geometric solutions to De l'Hospital's pulley problem, *College Math. J.* **30** (1999) 311–314.
3. Dijkstra Symposium: *In Pursuit of Simplicity*, University of Texas at Austin (May 2000); available at http://www.cs.utexas.edu/users/UTCS/festivities/dijkstra_symposium/.
4. E. W. Dijkstra, "How Do We Tell Truths That Might Hurt?", *EWD498* (1975); available at <http://www.cs.utexas.edu/users/EWD/transcriptions/EWD04xx/EWD498.html>.
5. W. H. J. Feijen, A. J. M. van Gasteren, D. Gries, and J. Misra, eds., *Beauty is Our Business*, Springer-Verlag, Berlin, 1990.
6. S. Gueron and R. Tessler, The Fermat-Steiner problem, this MONTHLY **109** (2002) 443–451.
7. N. Miller, The problem of a non-vanishing girder rounding a corner, this MONTHLY **56** (1949) 177–179.
8. C. Moretti, Moving a couch around a corner, *College Math. J.* **33** (2002) 196–200.
9. B. O'Toole, "Instantaneous Center of Rotation," available at http://www.me.unlv.edu/~bj/MEG_207/KoRB_inst_cent/index.htm.

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An Old Friend Revisited

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Let f_t be the real-valued function defined on the interval $(0, 1)$ that assigns 0 to each irrational number and $1/q^t$ to each rational p/q in lowest terms, where t is a real number larger than 1. Clearly, f_t is continuous exactly at the irrationals, and it is not hard to show that f_t is nowhere differentiable when $t \leq 2$. Darst and Taylor prove in their note "Differentiating Powers of an Old Friend" [1] that if $t > 2$ the function f_t is